SOLVING TWO-MOVE GAMES WITH PERFECT INFORMATION

George B. Dantzig

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1700 MAIN ST. • SANTA MONICA • CALIFORNIA

SUMMARY

A two-move game with perfect information is considered, such as a move and counter-move situation between two firms or economies. This leads to the problem of finding a global minimum of a concave function over a convex domain and the distressing possibility of local minima at every extreme point. It is shown however that the global minimum can be obtained by solving a linear programming system with side conditions that at least one of certain pairs of variables vanish. The latter problem can be shown to be equivalent to solving a linear programming problem with some integer valued variables.

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Consider a two-move game where player X can engage in any vector $x = (x_1, x_2, \ldots, x_n)$ of activity levels $x_j \geq 0$, consistent with a fixed inventory vector $e = (e_1, e_2, \ldots, e_m)$, say

$$(1) \qquad Ex = e \qquad (x \geq 0)$$

where E is an m x n matrix. This **constitutes X's move.** In so doing he leaves an inventory position $f + \bar{E}x$ for player Y where $\bar{E}$ is a given m' x n matrix and f an m' component vector. This requires that Y chose as his move an activity vector $y = (y_1, y_2, \ldots, y_{n'})$ so that

$$(2) \qquad Fy = f + \bar{E}x \qquad (y \geq 0)$$

where F is a given m' x n' matrix. It is assumed that x must be chosen so that an admissible move for Y exists. We remark in passing that a chess or checker game restricted to one move by each player can be cast in this form if there are added side constraints regarding the discrete character of a move. However a competitive situation of a move and a counter-move between two firms or two economies, would be more significant.

Let us suppose the payment to Y by X is given by

$$(3) \qquad z = \alpha x - \beta y$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$. It is clear that an optimum for X is to chose x so that his payment to Y is

(4) $$\hat{z} = \operatorname*{Min}_{x} \, [\alpha x - \operatorname*{Min}_{y|x} \beta y]$$

where we further assume βy is bounded from below for fixed x.

This is basically a very difficult problem because $\text{Min}_y \beta y$ for y satisfying (2) is a <u>convex function</u> of x but this implies that

(5) $$z' = [\alpha x - \text{Min}_y \, by]$$

is a <u>concave function</u> of x which is to be minimized over a convex domain of x satisfying (1) and (2). This can lead to local optima at one, many, or all extreme points of the convex domain of x.

For example suppose

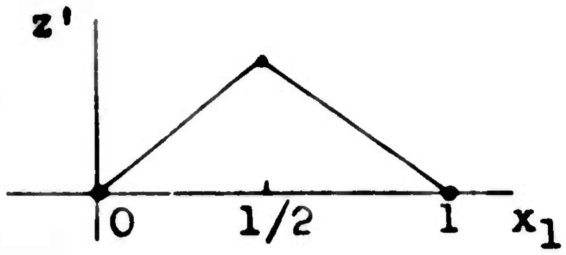
(6) $$\begin{array}{ll} x_1 \leq 1 & x_1 \geq 0 \\ y_1 \leq 1 - x_1 & y_1 \geq 0 \\ y_1 \leq x_1 & \\ z = 0 \cdot x_1 - (-y_1) = y_1 \, , & \end{array}$$

then the function z' to be minimized is

(7) $$z' = - \text{Min}(-y) = \begin{cases} x_1 & \text{if } 0 \leq x_1 \leq 1/2 \\ 1 - x_1 & \text{if } 1/2 \leq x_1 \leq 1 \end{cases}$$

which has two local minima, one at $x_1 = 0$ and the other at $x_1 = 1$:

(8)



The values of z' at these local minima happen to be equal but a slight perturbation could cause either one to be the global minimum.

By careful application of the duality theorem this problem can be reduced to a linear programming problem subject to a set of n' pairs of linear conditions either $y_j \geq 0$ or $\eta_j \geq 0$ for $j = 1,2,\ldots,n'$; here η_j are the dual variables along with $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$ satisfying

$$\pi F_j + \eta_j = \beta_j \qquad \eta_j \geq 0, (j=1,2,\ldots,n') \tag{9}$$

where F_j is the j^{th} column of F. We first remark for any fixed x, there exist an optimum $y = y^*$ satisfying (2) which minimizes βy. Associated with this x is also an optimum solution to the dual of (2) with variables π (unrestricted in sign associated with the m' equations) and non-negative variables $\eta_j \geq 0$ corresponding to y_j satisfying (9). The necessary and sufficient conditions that a solution of the primal and dual systems be optimal is that

(10) $\quad$ either $y_j = 0$ $\quad$ for $j = 1, 2, \ldots, n'$

$\quad$ or $\eta_j = 0$.

We now prove the following fundamental theorem:

THEOREM: An optimal solution to the two-move game (1), (2), (3) is found by choosing x and y satisfying (1) and (2), auxiliary variables π and η satisfying (9) and (10), and Min z satisfying (3).

Proof: The proof is along standard lines and immediate. An optimal solution to the game exists at one of the extreme points of the convex of x defined by (1) and (2) say at $x = \hat{x}$ for which there is a $y = \hat{y}$ and $\pi = \hat{\pi}$, $\eta = \hat{\eta}$ that satisfy (2), (9), (10) and yields the value $z = \hat{z}$ defined by (4). Hence

$$\text{Min } z \leq \hat{z} \tag{11}$$

On the other hand we can produce a solution x^*, y^*, π^*, η^* to (1), (2) (9), (10) which minimizes z by devices considered in [1] which shows that this type of problem is equivalent to a linear programming problem with some integer valued variables for which efficient procedure may exist [2], [3]. For the chosen value of $x = x^*$, (10) implies that the y^* is chosen so as to minimize βy. Hence the set of x^*, y^*, chosen this way is an admissible two moves in a game and its z = Min z must satisfy

$$\hat{z} \leq \text{Min } z \; ; \tag{12}$$

whence from (11) we have

$$(13) \qquad \hat{z} = \text{Min } z$$

completing the proof.

REFERENCES

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